ON THE (p, q)-TYPE AND BEST APPROXIMATION OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES IN BANACH SPACES

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ABSTRACT:
In this paper the characterizations of (p, q) −type of entire functions of two complex variables have been studied in terms of approximation errors. The results can be extended to m variables but to reduce the mechanical labor we have considered only two variables.

KEY WORDS: Order, type, entire function, index-pair, approximation error.

1. INTRODUCTION:

Let $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1}z_2^{m_2})\}$ be a function of the complex variables z_1 and z_2, regular for $|z_1| \le r_n$ $n = 1,2$. If r_1 and r_2 can be taken arbitrarily large, then $\varphi(z_{1.}z_2)$ represents an entire function of complex variables z_1 and z_2. Following Bose and Sharma [1] we define the maximum modulus of $\varphi(z_{1.}z_2)$ as

$M(r_1, r_2) = \max_{|z_n|\le r_n}|\varphi(z_{1.}z_2)|$, $n = 1,2.$

The order ρ of the entire function $\varphi(z_{1.}z_2)$ is defined as [1, P.219];

$$\rho = \lim_{r_1,r_2\to\infty} \sup \frac{\log\log M(r_1,r_2)}{\log(r_1r_2)}$$

For $0 < \rho < \infty$ the type τ of an entire function $\varphi(z_{1.}z_2)$ is defined as [1, p. 223] :

$$\tau = \lim_{r_1,r_2\to\infty} \sup \frac{\log M(r_1,r_2)}{r_1^{\rho}+r_2^{\rho}}.$$

if and only if

$$\mu = \lim_{m_1,m_2\to\infty} \sup \frac{\log({m_1}^{m_1}, {m_2}^{m_2})}{\log\left(\left|a_{m_1,m_2}\right|^{-1}\right)}$$

Bose and Sharma [1] obtained the following characterization for order and type of entire functions of two complex variables.

Theorem 1.1. The entire function $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1}z_2^{m_2})\}$ is of finite order

is finite and then the order ρ of $\varphi(z_{1.}z_2)$ is equal to μ. Define

$$\alpha = \lim_{m_1,m_2\to\infty} \sup \sqrt[(m_1+m_2)]{m_1^{m_1}m_2^{m_2}\left|a_{m_1,m_2}\right|^{\rho}}.$$

Recently, Ganti and Srivastava [2] characterized the order and type in terms of the approximation errors $E_{m_1,m_2(\varphi,B(u,v,k))}$ and $E_{m_1,m_2(\varphi,T_\vartheta)}$ But their results leave to study a big class of entire functions such as slow growth and fast growth. To bridge this gap in this paper we pick up a concept of (p, q) −type introduced by Juneja et al. [3] and consider it for entire functions of two variables. Roughly speaking, this concept is a modification of the classical definition of type obtained by replacing logarithms by iterated logarithms, where the degrees of iteration are determined by p and q .

To the best of our knowledge, characterizations for the (p,q) −type of entire functions of two complex variables in Banach spaces have not been obtained so far. In this paper, we have made an attempt to solve this problem.
We define the (p,q) −order and type of an entire function $\varphi(z_{1.}z_2)$ by

$$\rho(p,q) = \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} M(r_1,r_2)}{\log^{[q]}(r_1 r_2)}$$

and

$$\tau(p,q) = \lim_{r_1,r_2\to\infty} \sup \frac{\log^{[p]} M(r_1,r_2)}{(\log^{[q-1]} r_1)^{\rho(p,q)} + ((\log^{[q-1]} r_2)^{\rho(p,q)})}$$

where p and q are integers such that $p \geq q \geq 1$ and $0 \leq \tau(p,q) \leq \infty$.

2. Basic results:

In this section we have given some lemmas as basic results, which can be proved in a similar manner following the Kumar and Arora [4] and will be used in the sequel.

Lemma 2.1. If $\varphi(z_{1.}z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1} z_2^{m_2})\}$ be an entire function of index pair (p,q) if $0 < V < \infty$ .the function $\varphi(z_{1.}z_2)$ is of (p,q) −order $\rho(p,q)(b < \rho(p,q) < \infty)$ and (p,q) −type $T(p,q)$ if and only if $T(p,q) = MV$ where

$$V = V(p,q) = \lim_{m_1,m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\left[\log^{[q-1]}\left(\frac{1}{|a_{m_1,m_2}|}\right)^{\frac{1}{m_1+m_2}}\right]^{\rho(p,q)-A}} \qquad (2.1)$$

$$A = \begin{bmatrix} 1 & \text{if } (p,q) = (2,2) \\ 0 & \text{if } (p,q) \neq (2,2) \end{bmatrix}$$

$$M = M(p,q) \begin{cases} (\rho-1)^{\rho-1} & \text{if } (p,q) = (2,1) \\ \dfrac{1}{e^{\rho}} & \text{if } (p,q) = (2,2) \\ 1 & \text{if } p \geq 3 \end{cases}$$

$$b = \begin{bmatrix} 1 & \text{if } p = q \\ 0 & \text{if } p > q \end{bmatrix}$$

and

$$\begin{cases} \dfrac{(m_1^{m_1} m_2^{m_2})^{\frac{1}{m_1+m_2}}}{m_1+m_2} & ;\ m_1,m_2 \geq 1 \text{ for } (p,q) = (2,1) \\ 1 & ;\ m_1,m_2 \geq 1 \text{ for } 2 \leq q \leq p < \infty \\ 0 & ;\ \text{at teast one } m_1,m_2 = 0 \end{cases}$$

3. Main Results: In this section we prove our main results.

Theorem 3.1. Let $\varphi(z_1 z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1} z_2^{m_2})\}$ then the entire function $\varphi(z_1 z_2) \in \beta(u,v,k)$ of an entire function of index pair (p,q) if $0 < V^* < \infty$ the function $\varphi(z_1 z_2)$ is of (p,q) −order $\rho(p,q)(b < \rho(p,q) < \infty)$ and (p,q) −type if and only if $H(p,q) = MV^*$
where

$$V^* = V^*(p,q) = \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\left[\log^{[q-1]}\left(\frac{1}{E_{m_1,m_2}}(\varphi,\beta(u,v,k))\right)^{\frac{1}{m_1+m_2}}\right]^{\rho(p,q)-A}} \quad (3.1)$$

Proof. We prove the above result in two steps, first we consider the space $\beta(u,v,k)$ $v = 2$, $0 < u < 2$ *and* $k \geq 1$ .Let $\varphi(z_1 z_2) \in \beta(u,v,k)$ be of (p,q) −type $H(p,q)$ from (2.1), for any $\epsilon > 0$ there exists a natural number $m_0 = m_0(\varepsilon)$ such that

$$|a_{m_1,m_2}| \leq \exp\left\{-(m_1+m_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{V^*(p,q)+\varepsilon}\right)\right\}^{\frac{1}{\rho-A}} m_1, m_2 > m_0 \quad (3.2)$$

Now we denote the partial sum of the Taylor series of a function $\varphi(z_1 z_2)$ by

$$H_{m_1,m_2}(\varphi, z_1 z_2) = \sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} a_{i_1,i_2} z_1^{i_1} z_2^{i_2}$$

and

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) = \|\varphi - H_{m_1,m_2}(\varphi)\|_{u,2,k}$$

$$= \left\{\int_0^1\int_0^1 \{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\left(\sum_{i_1}\sum_{i_2} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2\right)^{\frac{k}{2}} dr_1 dr_2\right\}^{\frac{1}{k}}, \quad (3.3)$$

where

$$\sum_{i_1}\sum_{i_2} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2 = R_1 + R_2 + \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2,$$

$R_1 = \sum_{i_1=0}^{m_1}\sum_{i_2=m_2+1}^{\infty} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2$ and $R_2 = \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=0}^{m_2} r_1^{2i_1} r_2^{2i_2}|a_{i_1,i_2}|^2$
Since R_1, R_2 are bounded and $r_1, r_2 < 1$,therefore (3.3) becomes

$$E_{m_1,m_2}(\varphi,\beta(u,2,k)) \leq C\left[\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r^{(s+1)k} dr\right]\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^2\right\}^{\frac{1}{2}}$$

Where

$$\left[\int_0^1 \left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r^{(s+1)k} dr\right]$$

$$= \left[\int_0^1 \left\{(1-r_1)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r_1{}^{(m_1+1)k} dr_1\right] \times \left[\int_0^1 \left\{(1-r_2)^{k\left(\frac{1}{u}-\frac{1}{2}\right)-1}\right\} r_2{}^{(m_2+1)k} dr_2\right]$$

Therefore

$$E_{m_1,m_2}(\varphi, \beta(u,2,k)) \le C' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k) \left\{\sum_{i_1=m_1+1}^{\infty} \sum_{i_2=m_2+1}^{\infty} \left|a_{i_1,i_2}\right|^2\right\}^{\frac{1}{2}} \qquad (3.4)$$

where C' is a constant and $\beta(a,b)(a,b) > 0$ denotes the beta function.
In view of (3.2), we have

$$\sum_{i_1=m_1+1}^{\infty} \sum_{i_2=m_2+1}^{\infty} \left|a_{i_1,i_2}\right|^2$$

$$\le \sum_{i_1=m_1+1}^{\infty} \sum_{i_2=m_2+1}^{\infty} \left[\exp - (i_1+i_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(i_1+i_2)a_{i_1,i_2}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right]^2$$

Using the above inequality in (3.4) we get

$$E_{m_1,m_2}(\varphi, \beta(u,2,k)) \le C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k) \times$$

$$\left[\exp\left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]. (3.5)$$

The result for has been obtained by Ganti and Srivastava [2].
Now consider for (p , q) = (2 ,2)

$$E_{m_1,m_2}(\varphi, \beta(u,2,k)) \le C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k) \times$$

$$\left[\exp\left\{-(m_1+1+m_2+1)\left(\frac{\{(m_1+1+m_2+1)\}}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

Or

$$\log E_{m_1,m_2}(\varphi, \beta(u,2,k)) - \log\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k) \le$$

$$\left[\left\{-(m_1+1+m_2+1)\left(\frac{\{(m_1+1+m_2+1)\}}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

Or

$$\frac{1}{\log E_{m_1,m_2}(\varphi, \beta(u,2,k)) - \log\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k)}$$

$$\ge \frac{1}{-(m_1+1+m_2+1)\left(\frac{\{(m_1+1+m_2+1)\}}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}}$$

Since

$$\left\{\beta\left[(n+1)k+1; k\left(\frac{1}{u}-\frac{1}{2}\right)\right]\right\}^{1/(n+1)} \cong 1. \qquad (3.6)$$

Now for $(p, q) \neq (2,1)$ and $(2,2)$ we have from (3.5) that

$$E_{m_1,m_2}(\varphi, \beta(u,2,k)) \leq C'' \beta(m_{1,}u,2,k)\beta(m_{2,}u,2,k) \times$$
$$\left[\exp\left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

$$\Rightarrow \log E_{m_1,m_2}(\varphi, \beta(u,2,k)) - \log\beta(m_{1,}u,2,k) - \log\beta(m_{2,}u,2,k) \leq$$
$$\left[\left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

Taking (3.6) into account we get

$$\frac{1}{\log E_{m_1,m_2}(\varphi, \beta(u,2,k))} \geq \frac{1}{-(m_1+m_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}}$$

As for $(p, q) \neq (2,2)$ $A = 0$ and $a_{m_1,m_2} = 1$
so

$$\log E_{m_1,m_2}(\varphi, \beta(u,2,k)) \leq -(m_1+m_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+m_2)\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho}}$$

Or

$$\left[\log^{[q-2]}\left(\frac{1}{(m_1+m_2)}\log\left(\frac{1}{E_{m_1,m_2}(\varphi, \beta(u,2,k))}\right)\right)\right]^{\rho} \geq \frac{\log^{[p-2]}(m_1+m_2)}{V^*(p,q)+\varepsilon}$$

$$\Rightarrow V^*(p,q)+\varepsilon \geq \frac{\log^{[p-2]}(m_1+m_2)}{\left[\log^{[q-2]}\left(\frac{1}{(m_1+m_2)}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,2,k))}\right)\right)\right]^{\rho}}$$

Proceeding to limits we obtain

$$V^*(p,q) \geq \lim_{m_1+m_2\to\infty}\sup \frac{\log^{[p-2]}(m_1+m_2)}{\left[\log^{[q-2]}\left(\frac{1}{(m_1+m_2)}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,2,k))}\right)\right)\right]^{\rho}} \qquad (3.7)$$

Or

$$\frac{H(p,q)}{M} \geq \lim_{m_1+m_2\to\infty}\sup \frac{\log^{[p-2]}(m_1+m_2)}{\left[\log^{[q-2]}\left(\frac{1}{(m_1+m_2)}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,2,k))}\right)\right)\right]^{\rho}} \qquad (3.8)$$

To prove reverse inequality consider (Eq. 2.4 [2])

$$\log\left|a_{m_1,m_2}\right| + \log\beta(m_1,u,2,k) + \log\beta(m_2,u,2,k) \le \log E_{m_1,m_2}\left(\varphi,\beta(u,2,k)\right)$$

again (3.6) taking into account in above inequality we obtain

$$\lim_{m_1+m_2\to\infty}\sup\frac{m_1+m_2}{\left\{-\frac{1}{m_1+m_2}\log\left|a_{m_1,m_2}\right|\right\}^{\rho-1}} \le \lim_{m_1+m_2\to\infty}\sup\frac{m_1+m_2}{\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}\left(\varphi,\beta(u,2,k)\right)\right\}^{\rho-1}}$$

or

$$\lim_{m_1+m_2\to\infty}\sup\frac{m_1+m_2}{\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}\left(\varphi,\beta(u,2,k)\right)\right\}^{\rho(2,2)-1}} \ge V^*(2,2)$$

for (p , q) ≠ (2 ,1) and (2 ,2),we have

$$\lim_{m_1+m_2\to\infty}\sup\frac{\log^{(p-2)}m_1+m_2}{\log^{(q-2)}\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}\left(\varphi,\beta(u,2,k)\right)\right\}^{\rho}} \ge V^*(p,q)\,. \quad (3.9)$$

Hence combining above results we get the required result.
This is complete proof for the first step.
Now we consider the space $\beta(u,v,k), v \neq 2$, we have

$$E_{m_1,m_2}\left(\varphi,\beta(u,v,k)\right) = \left\|\varphi - H_{m_1,m_2}(\varphi)\right\|_{u,v,k}$$

$$= \left\{\int_0^1\int_0^1\{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\left(\sum_{i_1}\sum_{i_2} r_1^{vi_1}\, r_2^{vi_2}\left|a_{i_1,i_2}\right|^v\right)^{\frac{k}{v}} dr_1dr_2\right\}^{\frac{1}{k}} \quad (3.10)$$

Where

$$\textstyle\sum_{i_1}\sum_{i_2} r_1^{vi_1}\, r_2^{vi_2}\left|a_{i_1,i_2}\right|^v = R_1 + R_2 + \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty} r_1^{vi_1} r_2^{vi_2}\left|a_{i_1,i_2}\right|^v,$$

$R_1 = \sum_{i_1}\sum_{i_2} r_1^{vi_1}\, r_2^{vi_2}\left|a_{i_1,i_2}\right|^v$ and $R_2 = \sum_{i_1}\sum_{i_2} r_1^{vi_1}\, r_2^{vi_2}\left|a_{i_1,i_2}\right|^v$

Since R_1, R_2 are bounded and $r_1, r_2 < 1$,therefore (3.10) becomes

$$E_{m_1,m_2}\left(\varphi,\beta(u,v,k)\right) \le C''\left[\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\}r^{(s+1)k}dr\right]\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left|a_{i_1,i_2}\right|^v\right\}^{\frac{1}{v}}$$

Where

$$\left\{\int_0^1\left\{(1-r)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\}r^{(s+1)k}dr\right\} = \left\{\int_0^1\left\{(1-r_1)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\}{r_1}^{(m_1+1)k}dr_1\right\}\left\{\int_0^1\left\{(1-r_2)^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1}\right\}{r_2}^{(m_2+1)k}dr_2\right\}$$

Therefore

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C''\beta(m_1,u,v,k)\beta(m_2,u,v,k)\left\{\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^v\right\}^{\frac{1}{v}}, \quad (3.11)$$

Where C'' is constant and $\beta(m,u,v,k)$ is Euler's integral of the first kind. By using (3.2) we have

$$\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|^v$$

$$\le \sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}\left[\exp\left\{-(i_1+i_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(i_1+i_2)a_{i_1,i_2}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]^v$$

Using the above inequality in (3.11), we get

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C''\beta(m_1,u,v,k)\beta(m_2,u,v,k)\times$$

$$\left[\exp\left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]. \quad (3.12)$$

The result for $(p,q)=(2,1)$ has been obtained by Ganti and Srivastava [2]. Now consider for $(p,q)=(2,2)$

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \le C''\beta(m_1,u,v,k)\beta(m_2,u,v,k)\times$$

$$\left[\exp\left\{-(m_1+1+m_2+1)\left(\frac{(m_1+1+m_2+1)}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

$$\log E_{m_1,m_2}(\varphi,\beta(u,v,k)) - \log\beta(m_1,u,v,k) - \log\beta(m_2,u,v,k)$$

$$\le \left\{-(m_1+1+m_2+1)\left(\frac{(m_1+1+m_2+1)}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}$$

Or

$$\frac{1}{\log E_{m_1,m_2}(\varphi,\beta(u,v,k)) - \log\beta(m_1,u,v,k) - \log\beta(m_2,u,v,k)}$$

$$\ge -\frac{1}{(m_1+1+m_2+1)\left(\frac{(m_1+1+m_2+1)}{V^*(2,2)+\varepsilon}\right)^{\frac{1}{\rho-A}}}$$

Since

$$\beta\left[(n+1)k+1, k\left(\frac{1}{u}-\frac{1}{v}\right)\right] = \frac{\Gamma(n+1)(k+1)\Gamma\left(k\left(\frac{1}{u}-\frac{1}{v}\right)\right)}{\Gamma\left((n+1)(k+1)+k\left(\frac{1}{u}-\frac{1}{v}\right)\right)}$$

and

$$\beta\left[(n+1)k+1, k\left(\frac{1}{u}-\frac{1}{v}\right)\right]^{\frac{1}{n+1}} \cong 1. \quad (3.13)$$

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Hence proceeding to limits, we get

$$V^*(2,2) \geq \lim_{m_1+m_2\to\infty}\sup \frac{(m_1+m_2)}{\left[\frac{1}{(m_1+m_2)}\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right]^{\rho(2,2)-1}}$$

$$\frac{H(2,2)}{M} \geq \lim_{m_1+m_2\to\infty}\sup \frac{(m_1+m_2)}{\left[\frac{1}{(m_1+m_2)}\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right]^{\rho(2,2)-1}} \quad (3.14)$$

Now for $(p,q) \neq (2,1)$ and $(2,2)$ we have from (3.12) ,we get

$$E_{m_1,m_2}(\varphi,\beta(u,v,k)) \leq C''\beta(m_1,u,v,k)\beta(m_2,u,v,k) \times$$

$$\left[\exp\left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}\right\}\right]$$

Or

$$\log E_{m_1,m_2}(\varphi,\beta(u,v,k)) - \log\beta(m_1,u,v,k) - \log\beta(m_2,u,v,k)$$

$$\leq -(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)a_{m_1+1,m_2+1}\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}$$

Using (3.13) ,we obtain

$$\frac{1}{\log E_{m_1,m_2}(\varphi,\beta(u,v,k))} \geq -\frac{1}{(m_1+m_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}(m_1+m_2)}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho-A}}}$$

for $(p,q) \neq (2,1)$ $A = 0$ and $a_{m_1,m_2}=1$

so we have

$$\log E_{m_1,m_2}(\varphi,\beta(u,v,k)) \leq -(m_1+m_2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}(m_1+m_2)}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho}}$$

Or

$$\left[\log^{(q-2)}\left(\frac{1}{m_1+m_2}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right)\right]^{\rho} \geq \frac{\log^{[p-2](m_1+m_2)}}{V^*(p,q)+\varepsilon}$$

Or

$$V^*(p,q)+\varepsilon \geq \frac{\log^{[p-2](m_1+m_2)}}{\left[\log^{(q-2)}\left(\frac{1}{m_1+m_2}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right)\right]^{\rho}}$$

Now proceeding to limits, we obtain

$$V^*(p,q) \geq \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}(m_1+m_2)}{\left[\log^{(q-2)}\left(\frac{1}{m_1+m_2}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right)\right]^{\rho}}$$

Or

$$\frac{H(p,q)}{M} \geq \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}(m_1+m_2)}{\left[\log^{(q-2)}\left(\frac{1}{m_1+m_2}\log\left(\frac{1}{E_{m_1,m_2}(\varphi,\beta(u,v,k))}\right)\right)\right]^{\rho}} \quad (3.15)$$

To prove reverse inequality taking (3.11) into account which gives

$$\left|a_{m_1,m_2}\right|\left[\beta(m_{1,}u,v,k)+\beta(m_{2,}u,v,k)\right] \leq E_{m_1,m_2}(\varphi,\beta(u,v,k))$$

Or

given using (3.13) in above inequality ,we obtain

$$\lim_{m_1+m_2\to\infty} \sup \frac{(m_1+m_2)}{\left\{-\frac{1}{m_1+m_2}\log\left|a_{m_1,m_2}\right|\right\}^{\rho-1}} \leq \lim_{m_1+m_2\to\infty} \sup \frac{(m_1+m_2)}{\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}(\varphi,\beta(u,v,k))\right\}^{\rho-1}}$$

Or

$$\lim_{m_1+m_2\to\infty} \sup \frac{(m_1+m_2)}{\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}(\varphi,\beta(u,v,k))\right\}^{\rho-1}} \geq V^*(2,2) \quad (3.16)$$

for $(p,q) \neq (2,1)$ and $(2,2)$ we have

$$\lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}(m_1+m_2)}{\log^{[q-2]}\left\{-\frac{1}{m_1+m_2}\log E_{m_1,m_2}(\varphi,\beta(u,v,k))\right\}^{\rho-1}} \geq V^*(p,q) \quad (3.17)$$

Combining (3.14), (3.15), (3.16) and (3.17) the proof of second step is immediate.
Now we consider third step. Let $0 < u < v < 2$ and $k, v \geq 1$.
Since

$$E_{m_1,m_2}\left(\varphi,\beta(u_{1,}v_{1,}k_1)\right) \leq 2^{\left(\frac{1}{u_{1,}}-\frac{1}{v_{1,}}\right)}\left[k\left(\left(\frac{1}{u}-\frac{1}{v}\right)\right)\right]^{\left(\frac{1}{u_{1,}}-\frac{1}{v_{1,}}\right)} E_{m_1,m_2}(\varphi,\beta(u,v,k))$$

where $u_1 = u, v_1 = 2$ and $k_1 = k$ and the condition (3.1) is already proved for the space $\beta(u,2,k)$ hence

$$\lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\{\log E_{m_1,m_2}(\varphi,\beta(u,v,k))\}^{\frac{1}{m_1+m_2}}} \geq \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\{\log E_{m_1,m_2}(\varphi,\beta(u,2,k))\}^{\frac{1}{m_1+m_2}}} \quad (3.18)$$

Now let $0 < u \le 2 < v$ since

$$M_1(\varphi, r_1, r_2) \le M_2(\varphi, r_1, r_2) \quad 0 < r_1 < r_2 < 1$$

therefore

$$E_{m_1,m_2}(\varphi, \beta(u,v,k)) \ge \left[\left\{\int_0^1 \int_0^1 \{(1-r_1)(1-r_2)\}^{k\left(\frac{1}{u}-\frac{1}{v}\right)-1} Q\, dr_1 dr_2\right\}\right]^{\frac{1}{k}}$$
$$\ge \left[|a_{m_1,m_2}|\beta(m_1, u, v, k)\beta(m_2, u, v, k)\right]$$

where $Q = \inf[M_2^k(\varphi - u, r_1, r_2) : p \in P]$. Hence we get

$$\lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\{\log E_{m_1,m_2}(\varphi, \beta(u,v,k))\}^{\frac{1}{m_1+m_2}}}$$
$$\ge \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\left\{\frac{1}{|a_{m_1,m_2}|}\right\}^{\frac{1}{m_1+m_2}}} \qquad (3.19)$$

In view of (3.18), (3.19) and Lemma 2.1 we get the required result. This completes the proof of the theorem.

Theorem 3.2. If $\varphi(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1} z_2^{m_2})\}$ be an entire function, then for a pair of integers $(p,q), p \ge 2, q \ge 1$ the function $\varphi(z_1, z_2) \in H_v$ is of (p,q) −order ρ is of type τ if and only if $\tau(p,q) = MV^*$

$$V^*(p,q) = \lim_{m_1+m_2\to\infty} \sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\left[\left\{\frac{1}{E_{m_1,m_2}(\varphi, H_v)}\right\}^{\frac{1}{m_1+m_2}}\right]^{\rho-A}} \quad . \qquad (3.20)$$

Proof. Let $\varphi(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty}\{a_{m_1,m_2}(z_1^{m_1} z_2^{m_2})\} \in H_v$ be an entire transcendental function. Since φ is entire, we have

$$\lim_{m_1+m_2\to\infty} \sqrt[(m_1+m_2)]{a_{m_1,m_2}} = 0 \qquad (3.21)$$

and $\varphi(z_1, z_2) \in H_v$ therefore

$$M_v(\varphi, r_1, r_2) < \infty$$

and $\varphi(z_1, z_2) \in \beta(u,v,k), 0 < u < v \le \infty; v, k \ge 1$. By (1.1) we have

$$E_{m_1,m_2}\left(\varphi, \beta\left(\frac{v}{2}, v, v\right)\right) \le K_v E_{m_1,m_2}(\varphi, H_v) \; 1 \le v < \infty \qquad (3.22)$$

where K_v is a constant independent of m_1, m_2 and φ.
In the case of space H_∞.

$$E_{m_1,m_2}(\varphi, \beta(u, \infty, \infty)) \le E_{m_1,m_2}(\varphi, H_v), 0 < u < \infty. \qquad (3.23)$$

From (3.22) we have

$$\xi(\varphi) = \lim_{m_1+m_2\to\infty}\sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\left[\left\{\frac{1}{E_{m_1,m_2}(\varphi,H_v)}\right\}^{\frac{1}{m_1+m_2}}\right]^{\rho-A}}$$

$$\geq \lim_{m_1+m_2\to\infty}\sup \frac{\log^{[p-2]}\{(m_1+m_2)a_{m_1,m_2}\}}{\log^{[q-1]}\left[\left\{\frac{1}{E_{m_1,m_2}\left(\varphi,\beta\left(\frac{v}{2},v,v\right)\right)}\right\}^{\frac{1}{m_1+m_2}}\right]^{\rho-A}} \qquad (3.24)$$

for $1 \leq v < \infty$ using (3.23) we prove inequality (3.24).
For reverse inequality

$$E_{m_1,m_2}(\varphi,H_v) \leq o(1)\sum_{i_1=m_1+1}^{\infty}\sum_{i_2=m_2+1}^{\infty}|a_{i_1,i_2}|$$

Using (3.2) we have

$$E_{m_1,m_2}(\varphi,H_v) \leq o(1)\exp\left\{-(m_1+m_2+2)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+m_2+2)\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho}}\right\}$$

Or

$$\log E_{m_1,m_2}(\varphi,H_v) \leq \left\{-(m_1+1+m_2+1)\exp^{[q-2]}\left(\frac{\log^{[p-2]}\{(m_1+1+m_2+1)\}}{V^*(p,q)+\varepsilon}\right)^{\frac{1}{\rho}}\right\}$$

Or

$$V^*(p,q)+\varepsilon \geq \frac{\log^{[p-2]}\{(m_1+1+m_2+1)\}}{\left[\log^{[q-2]}\left(-\frac{\log E_{m_1,m_2}(\varphi,H_v)}{(m_1+1+m_2+1)}\right)\right]^{\rho}}$$

Proceeding to limits

$$V^*(p,q) \geq \frac{\log^{[p-2]}\{(m_1+1+m_2+1)\}}{\left[\log^{[q-2]}\left(-\frac{\log E_{m_1,m_2}(\varphi,H_v)}{(m_1+1+m_2+1)}\right)\right]^{\rho}}. \qquad (3.25)$$

In the consequence of Theorem 3.1 with (3.24) and (3.25) we obtain the result immediately.
Now to prove sufficiency, assume that the condition (3.20) is satisfied. Then it follows that

$$E^{\rho}_{m_1,m_2} \to 0 \text{ as } m_1, m_2 \to \infty.$$

This yield

$$\lim_{m_1+m_2\to\infty} \sqrt[(m_1+m_2)]{E_{m_1,m_2}(\varphi,H_v)} = 0.$$

This relation and the estimate $|a_{m_1,m_2}(\varphi)| \leq E_{m_1,m_2}(\varphi,H_v)$ yield the inequality (3.21). This implies that $\varphi(z_1,z_2) \in H_v$ is an entire transcendental function.

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